Chapter 9

Least-Squares Finite Element Solution of Compressible Euler Equations

There are a number of fundamental differences between the numerical solution of incompressible and compressible flows. The first one is the necessity of using an equation of state (EOS) for compressible flows. For incompressible flows no EOS exist, but for incompressible flows we typically work with the following ideal gas EOS

$$p = \rho RT \tag{9.1}$$

which relates the pressure, density and temperature of a gas. Air is the most commonly used working fluid for compressible flow applications, for which the gas constant is $R = 287.1\ J/kgK$.

The second major difference is the role of pressure. For incompressible flows pressure has no thermodynamic meaning, it just adjusts itself so that the velocity field remains incompressible at all times. But for compressible flows pressure is a thermodynamic variable related to other termodynamic variables through an EOS.

Third difference is about the number of unknowns and equations. For a 3D, isothermal, incompressible flow the unknowns are three velocity components and pressure. These four scalars can be solved by the use of three scalar momentum equations and the continuity equation. If the flow is not isothermal then the temperature appears in the unknown list and the energy equation needs to be solved too, but except for natural convection problems energy equation is decoupled from the other equations and can be solved separately after obtaining the velocity field. But for compressible flows density and temperature are always in the unknown list. For a 3D, compressible flow there are 6 unknowns (three velocity components, density, pressure and temperature) which can be solved using 6 equations (three scalar momentum equations, continuity equation, energy equation and EOS). All these 6 equations are coupled and need to be solved together.

Fourth difference is related to the numerical challenges and solution techniques that can be used. For incompressible flows pressure does not appear in the continuity equation and it is said that there is a weak coupling between velocity and pressure unknowns. LBB compatibility condition, related numerical instabilities and the use of stabilization techniques are mainly due to this weak coupling. However, for compressible flows density, which is related to pressure through the EOS, appears in the continuity equation and there is a stronger coupling between the unknowns. LBB compatibility condition does not exist for compressible flows, therefore we are not restricted to work with LBB-stable elements. Altough, the use of artificial vicosity is common for the solution of Euler equations, residual based stabilization techniques such as SUPG or GLS are rarely used for compressible flow solutions. Compressible flows are almost always solved using the unsteady forms of the equations. Commonly used FE variants for compressible flows are Taylor-Galerkin FEM, Least-Squares FEM.

There are also important concepts like Flux Corrected Transport (FCT) and Total Variation Diminishing (TVD), which are only used for compressible flow solutions.

9.1 Compressible Euler Equations

Although inviscid Euler equations are rarely used for incompressible flow solutions, they are frequently used to simulate compressible flows. Especially for external aerodynamic flows over streamlined bodies such as wings, pressure distribution and the corresponding lift and drag coefficients are the main interest and Euler equations are capable of prediciting these for certain flow conditions. Euler equations should be used with care when viscous affects are important for problems such as internal flows and external aerodynamic flows with high angle of attack where flow separation can occur.

In the context of numerical solution of compressible flows, the term "Euler equations" is used for the whole equation set including the consevation of mass, linear momentum and energy as given below

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0 \qquad (9.2a)$$

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} + \frac{1}{\rho} \nabla p = 0 \qquad (9.2b)$$

$$\frac{\partial p}{\partial t} + \vec{V} \cdot \nabla p + \gamma p \left(\nabla \cdot \vec{V}\right) = 0 \qquad (9.2c)$$

where γ is the constant specific heat ratio, which is equal to 1.4 for air. In writing the energy equation (9.2c) ideal gas EOS is already used and for a 3D flow equation set (9.2) contains 5 scalar equations for 5 scalar unknowns (density, 3 velocity components and pressure). Equation set (9.2) takes the following form when written for a 2D flow on the *xy*-plane of the Cartesian coordinate system

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} = 0 \qquad (9.3a)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \qquad (9.3b)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \qquad (9.3c)$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \gamma p \frac{\partial u}{\partial x} + \gamma p \frac{\partial v}{\partial y} = 0 \qquad (9.3d)$$

where the scalar unknowns are ρ, u, v and p. These unknowns can be combined into the following unknown vector

$$U = \begin{Bmatrix} \rho \\ u \\ v \\ p \end{Bmatrix} \tag{9.4}$$

Using this unknown vector equation set (9.3) can be written in the following compact form

$$\frac{\partial U}{\partial t} + A_1 \frac{\partial U}{\partial x} + A_2 \frac{\partial U}{\partial y} = 0 \tag{9.5}$$

where

$$A_1 = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & 1/\rho \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{bmatrix} \quad , \quad A_2 = \begin{bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/\rho \\ 0 & 0 & \gamma p & v \end{bmatrix} \tag{9.6}$$

9.2 Time Discretization

In chapter 5 we studied semi-discrete formulation of unsteady problems. We'll now follow a slightly different route and perform time discretization first, followed by space discretization. Let's begin by introducing the following ΔU vector

$$\Delta U = U_{s+1} - U_s \qquad \text{or} \qquad \begin{bmatrix} \Delta\rho \\ \Delta u \\ \Delta v \\ \Delta p \end{bmatrix} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}_{s+1} - \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}_s \tag{9.7}$$

where U_s is the known solution at time level s and U_{s+1} is the unknown vector at the new time level $s+1$. Instead of solving for U_{s+1} directly, in this chapter <u>we'll prefer to solve for ΔU</u>.

Continuity equation (9.3a) written for the new time level $s+1$ is

$$\frac{\partial \rho_{s+1}}{\partial t} + \rho_{s+1}\frac{\partial u_{s+1}}{\partial x} + u_{s+1}\frac{\partial \rho_{s+1}}{\partial x} + \rho_{s+1}\frac{\partial v_{s+1}}{\partial y} + v_{s+1}\frac{\partial \rho_{s+1}}{\partial y} = 0 \tag{9.8}$$

The time derivative term of the above equation can be discretized using <u>backward differencing</u> as follows

$$\frac{\partial \rho_{s+1}}{\partial t} = \frac{\Delta\rho}{\Delta t} \tag{9.9}$$

Using Eqn (9.7) in (9.8), unknowns of Eqn (9.8) at time level $s+1$ can be expressed in terms of the known solution at level s and the difference between the two time levels. New continuity equation becomes

$$\frac{\Delta\rho}{\Delta t} + (\rho_s + \Delta\rho)\frac{\partial(u_s + \Delta u)}{\partial x} + (u_s + \Delta u)\frac{\partial(\rho_s + \Delta\rho)}{\partial x} +$$

$$(\rho_s + \Delta\rho)\frac{\partial(v_s + \Delta v)}{\partial y} + (v_s + \Delta v)\frac{\partial(\rho_s + \Delta\rho)}{\partial y} = 0 \tag{9.10}$$

Neglecting the following higher order, i.e. small terms in the above equation

$$\Delta u \frac{\partial \Delta \rho}{\partial x}, \qquad \Delta \rho \frac{\partial \Delta u}{\partial x}, \qquad \Delta v \frac{\partial \Delta \rho}{\partial y}, \qquad \Delta \rho \frac{\partial \Delta v}{\partial y}$$

it simplies to

$$\frac{\Delta \rho}{\Delta t} + \Delta \rho \frac{\partial u_s}{\partial x} + \rho_s \frac{\partial \Delta u}{\partial x} + \Delta u \frac{\partial \rho_s}{\partial x} + u_s \frac{\partial \Delta \rho}{\partial x} + \Delta \rho \frac{\partial v_s}{\partial y} + \rho_s \frac{\partial \Delta v}{\partial y} + \Delta v \frac{\partial \rho_s}{\partial y} + v_s \frac{\partial \Delta \rho}{\partial y} = -\left(\rho_s \frac{\partial u_s}{\partial x} + u_s \frac{\partial \rho_s}{\partial x} + \rho_s \frac{\partial v_s}{\partial y} + v_s \frac{\partial \rho_s}{\partial y}\right) \qquad (9.11a)$$

As seen the unknown differences (Δu, Δv and $\Delta \rho$) are on the left hand side of the equation and the right hand side contains only known values of time level s.

A similar procedure can be applied to the x-momentum equation (9.3b), y-momentum equation (9.3c) and the energy equation (9.3d) to discretize the time deriative using backward differencing and introduce Δ terms as the new unknowns. The resulting equations are

$$\frac{\Delta u}{\Delta t} + \Delta u \frac{\partial u_s}{\partial x} + u_s \frac{\partial \Delta u}{\partial x} + \Delta v \frac{\partial u_s}{\partial y} + v_s \frac{\partial \Delta u}{\partial y} - \Delta \rho \frac{1}{(\rho_s)^2} \frac{\partial p_s}{\partial x} + \frac{1}{\rho_s} \frac{\partial \Delta p}{\partial x} = -\left(u_s \frac{\partial u_s}{\partial x} + v_s \frac{\partial u_s}{\partial y} + \frac{1}{\rho_s} \frac{\partial p_s}{\partial x}\right) \qquad (9.11b)$$

$$\frac{\Delta v}{\Delta t} + \Delta u \frac{\partial v_s}{\partial x} + u_s \frac{\partial \Delta v}{\partial x} + \Delta v \frac{\partial v_s}{\partial y} + v_s \frac{\partial \Delta v}{\partial y} - \Delta \rho \frac{1}{(\rho_s)^2} \frac{\partial p_s}{\partial y} + \frac{1}{\rho_s} \frac{\partial \Delta p}{\partial y} = -\left(u_s \frac{\partial v_s}{\partial x} + v_s \frac{\partial v_s}{\partial y} + \frac{1}{\rho_s} \frac{\partial p_s}{\partial y}\right) \qquad (9.11c)$$

$$\frac{\Delta p}{\Delta x} + \gamma \Delta p \frac{\partial u_s}{\partial x} + \gamma p_s \frac{\partial \Delta u}{\partial x} + \Delta u \frac{\partial p_s}{\partial x} + u_s \frac{\partial \Delta p}{\partial x} + \gamma \Delta p \frac{\partial v_s}{\partial y} + \gamma p_s \frac{\partial \Delta v}{\partial y} + \Delta v \frac{\partial p_s}{\partial y} + v_s \frac{\partial \Delta p}{\partial y} = -\left(\gamma p_s \frac{\partial u_s}{\partial x} + u_s \frac{\partial p_s}{\partial x} + \gamma p_s \frac{\partial v_s}{\partial y} + v_s \frac{\partial p_s}{\partial y}\right) \qquad (9.11d)$$

This equation set (9.11) with the unknown vector ΔU can be put in a compact form similar to Eqn (9.5) as follows

$$A \Delta U = f \qquad \text{where} \qquad A = A_0 + A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} \qquad (9.12)$$

with A_1 and A_2 matrices are given in Eqn (9.6) and the new A_0 matrix and f vector are as follows

$$A_0 = \begin{bmatrix} \frac{1}{\Delta t}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} & \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & 0 \\ -\frac{1}{\rho^2}\frac{\partial p}{\partial x} & \frac{1}{\Delta t}+\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \\ -\frac{1}{\rho^2}\frac{\partial p}{\partial y} & \frac{\partial v}{\partial y} & \frac{1}{\Delta t}+\frac{\partial v}{\partial y} & 0 \\ 0 & \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{1}{\Delta t}+\gamma\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \end{bmatrix}$$

$$f = -\begin{bmatrix} \rho\frac{\partial u}{\partial x}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial y}+v\frac{\partial \rho}{\partial x} \\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \gamma p\frac{\partial u}{\partial x}+u\frac{\partial p}{\partial x}+\gamma p\frac{\partial v}{\partial y}+v\frac{\partial p}{\partial y} \end{bmatrix}$$

Note that for simplicity all the s subscripts are removed from A and f.

9.3 Introduction to Least-Squares Finite Element Method (LSFEM) Using a 1D Model DE

Eqn (9.12) is ready to be discretized in space, which will be done using Least-Squares FEM (LSFEM), instead of Galerkin FEM. In this section the following simple 1D model DE will be used to explain how LSFEM formulation works.

$$a\frac{du}{dx}+cu=f \tag{9.13}$$

Weighted integral statement written for a single element is

$$\int_{\Omega^e} w(x)\,R(x)\,dx=0 \tag{9.14}$$

where the residual is

$$R(x) \;=\; a\frac{du^e}{dx}+cu^e-f \tag{9.15}$$

with the following typical approximate solution over an element e

$$u^e(x) \;=\; \sum_{j=1}^{NEN} S_j^e(x)\,u_j^e \tag{9.16}$$

Main difference between Galerkin FEM and LSFEM is the selection of weight functions. In LSFEM the <u>i^{th} weight function is selected to be the derivative of the residual with respect to the i^{th} nodal unknown</u>. This selection also corresponds to the minimization of the integral of the square of the residual, and that's why the method is called "Least Squares". To obtain the i[th] weight function let's first substitute Eqn (9.16) into (9.15) to get

$$R(x) = a \sum_{j=1}^{NEN} u_j^e \frac{dS_j^e}{dx} + c \sum_{j=1}^{NEN} u_j^e S_j^e - f \qquad (9.17)$$

and take the derivative of this residual with respect to u_i^e

$$w_i = \frac{\partial R}{\partial u_i^e} = a \frac{dS_i^e}{dx} + c S_i^e \qquad (9.18)$$

Using this i[th] weight function in Eqn (9.14), the i[th] equation for element e becomes

$$\int_{\Omega^e} \left(a \frac{dS_i^e}{dx} + c S_i^e \right) \left(a \sum_{j=1}^{NEN} u_j^e \frac{dS_j^e}{dx} + c \sum_{j=1}^{N} u_j^e S_j^e - f \right) dx = 0 \qquad (9.19)$$

which can be arranged as follows

$$\sum_{j=1}^{NEN} \left[\int_{\Omega^e} \left(a \frac{dS_i^e}{dx} + c S_i^e \right) \left(a \frac{dS_j^e}{dx} + c S_j^e \right) dx \right] u_j^e = \int_{\Omega^e} f \left(a \frac{dS_i^e}{dx} + c S_i^e \right) dx \qquad (9.20)$$

Therefore the elemental stiffness matrix and elemental force vector are defined as

$$K_{ij}^e = \int_{\Omega^e} \left(a \frac{dS_i^e}{dx} + c S_i^e \right) \left(a \frac{dS_j^e}{dx} + c S_j^e \right) dx \qquad (9.21)$$

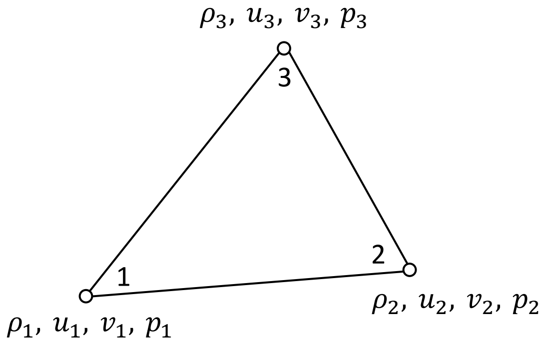
$$F_i^e = \int_{\Omega^e} f \left(a \frac{dS_i^e}{dx} + c S_i^e \right) dx \qquad (9.22)$$

In LSFEM, elemental stiffness matrix K^e is always symmetric, independent of the DE, which is an important advantage of the technique.

9.4 LSFEM Formulation of Euler Equations

Now we can apply the LSFEM technique that we learned in the previous section to the compressible Euler equations given by Eqn (9.12). Elemental stiffness matrix and force vector are given by Eqn (9.23). The important difference is that now we are not solving a single equation, but a set of equations and 4 unknowns are stored at each node of the FE mesh.

Consider the following 3-node triangular element, with three unknowns at each node (NNU=3). For this element there are 12 unknowns and the elemental stiffness matrix will be 12x12 as seen below. It is composed of 4 sub matrices of size 3x12 each. First submatrix comes from the continuity eqn, second one is from x-momentum equation, etc.



Number of Element Nodes: NEN = 3

Number of Nodal Unknowns: NNU = 4

Number of Elemental Unknowns: NEU = 12

Figure 9.1 Three-node triangular element with 12 unknowns

$$\begin{bmatrix} & [K^1] & \\ & [K^2] & \\ & [K^3] & \\ & [K^4] & \end{bmatrix}^e \begin{Bmatrix} \begin{Bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \end{Bmatrix} \\ \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{Bmatrix} \\ \begin{Bmatrix} \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \end{Bmatrix} \\ \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \end{Bmatrix} \end{Bmatrix}^e = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \end{Bmatrix}^e \tag{9.23}$$

Approximate solution over the element can be expressed as follows (Note that according to our previous formulation, unknowns are not the variables at the new time level but they are the difference of variables between the new and old time levels)

$$\Delta\rho^e = \sum_{j=1}^{NEN} \Delta\rho_j^e\ S_j \tag{9.24a}$$

$$\Delta u^e = \sum_{j=1}^{NEN} \Delta u_j^e\ S_j \tag{9.24b}$$

$$\Delta v^e = \sum_{j=1}^{NEN} \Delta v_j^e\ S_j \tag{9.24c}$$

$$\Delta p^e = \sum_{j=1}^{NEN} \Delta p_j^e\ S_j \tag{9.24d}$$

Using these approximations in the continuity equation (9.11a) we get (superscript e is dropped from the nodal unknowns for simplicty)

$$\frac{1}{\Delta t}\left(\sum_{j=1}^{NEN}\Delta\rho_j\,S_j\right)+\left(\sum_{j=1}^{NEN}\Delta\rho_j\,S_j\right)\frac{\partial u_s}{\partial x}+\rho_s\left(\sum_{j=1}^{NEN}\Delta u_j\,\frac{\partial S_j}{\partial x}\right)+\left(\sum_{j=1}^{NEN}\Delta u_j\,S_j\right)\frac{\partial \rho_s}{\partial x}$$
$$+u_s\left(\sum_{j=1}^{NEN}\Delta\rho_j\,\frac{\partial S_j}{\partial x}\right)+\left(\sum_{j=1}^{NEN}\Delta\rho_j\,S_j\right)\frac{\partial v_s}{\partial y}+\rho_s\left(\sum_{j=1}^{NEN}\Delta v_j\,\frac{\partial S_j}{\partial y}\right)+\left(\sum_{j=1}^{NEN}\Delta v_j\,S_j\right)\frac{\partial \rho_s}{\partial y}$$
$$+v_s\left(\sum_{j=1}^{NEN}\Delta\rho_j\,\frac{\partial S_j}{\partial y}\right)=\underbrace{-\left(\rho_s\frac{\partial u_s}{\partial x}+u_s\frac{\partial \rho_s}{\partial x}+\rho_s\frac{\partial v_s}{\partial y}+v_s\frac{\partial \rho_s}{\partial y}\right)}_{f_1} \quad (9.25)$$

where the nodal unknowns are shown in red. By taking the derivative of the residual of this equation with respect to the 1st , 2nd and 3rd unknowns, i.e. $\Delta\rho_1$, $\Delta\rho_2$ and $\Delta\rho_3$, we get the following first three weight functions

$$w_1=\frac{1}{\Delta t}S_1+S_1\frac{\partial u_s}{\partial x}+u_s\frac{\partial S_1}{\partial x}+S_1\frac{\partial v_s}{\partial y}+v_s\frac{\partial S_1}{\partial y}$$

$$w_2=\frac{1}{\Delta t}S_2+S_2\frac{\partial u_s}{\partial x}+u_s\frac{\partial S_2}{\partial x}+S_2\frac{\partial v_s}{\partial y}+v_s\frac{\partial S_2}{\partial y}$$

$$w_3=\frac{1}{\Delta t}S_3+S_3\frac{\partial u_s}{\partial x}+u_s\frac{\partial S_3}{\partial x}+S_3\frac{\partial v_s}{\partial y}+v_s\frac{\partial S_3}{\partial y}$$

Multipliying $w_1$with the residual of Eqn (9.25) and integrating over element e, we get the coefficients of the 1st row of Eqn (9.23). For example the very first entry of the elemental stiffness matrix corresponds to the terms multiplied by $\Delta\rho_1$ and it is the calculated as

$$K_{11}^{1\,e}=\int_{\Omega^e}w_1\left(\frac{1}{\Delta t}S_1+S_1\frac{\partial u_s}{\partial x}+u_s\frac{\partial S_1}{\partial x}+S_1\frac{\partial v_s}{\partial y}+v_s\frac{\partial S_1}{\partial y}\right)d\Omega$$

Or the terms multiplied by Δu_1, which is the fourth unknown, gives the following coefficient

$$K_{14}^{1\,e}=\int_{\Omega^e}w_1\left(\rho_s\frac{\partial S_1}{\partial x}+S_1\frac{\partial \rho_s}{\partial x}\right)d\Omega$$

Rigth hand side of the 1st row is obtained as

$$F_1^{1\,e}=\int_{\Omega^e}w_1\,f_1\,d\Omega$$

where f_1 is the right hand side of Eqn (9.25).

To get the entries of the 2nd and 3rd rows of Eqn (9.23) w_2 and w_3 are multipled with Eqn (9.25).

Entries of the 4th, 5th and 6th row of the elemental system comes from the x-momentum equation (9.11b), which can be written as follows by substituting Eqn (9.24) in it

$$\frac{1}{\Delta t}\left(\sum_{j=1}^{NEN} \Delta u_j \, S_j\right) + \left(\sum_{j=1}^{NEN} \Delta u_j \, S_j\right)\frac{\partial u_s}{\partial x} + u_s\left(\sum_{j=1}^{NEN} \Delta u_j \, \frac{\partial S_j}{\partial x}\right) + \left(\sum_{j=1}^{NEN} \Delta v_j \, S_j\right)\frac{\partial u_s}{\partial y} + v_s\left(\sum_{j=1}^{NEN} \Delta u_j \, \frac{\partial S_j}{\partial y}\right)$$
$$-\left(\sum_{j=1}^{NEN} \Delta \rho_j \, S_j\right)\frac{1}{(\rho_s)^2}\frac{\partial p_s}{\partial x} + \frac{1}{\rho_s}\left(\sum_{j=1}^{NEN} \Delta p_j \, \frac{\partial S_j}{\partial x}\right)$$
$$= -\left(u_s \frac{\partial u_s}{\partial x} + v_s \frac{\partial u_s}{\partial y} + \frac{1}{\rho_s}\frac{\partial p_s}{\partial x}\right) \tag{9.26}$$

Taking the derivative of the residual of Eqn (9.26) with respect to the 4th, 5th and 6th elemental unknowns, i.e. Δu_1, Δu_2 and Δu_3 we get the following 4th, 5th and 6th weight functions

$$w_4 = \frac{1}{\Delta t}S_1 + S_1\frac{\partial u_s}{\partial x} + u_s\frac{\partial S_1}{\partial x} + v_s\frac{\partial S_1}{\partial y}$$

$$w_5 = \frac{1}{\Delta t}S_2 + S_2\frac{\partial u_s}{\partial x} + u_s\frac{\partial S_2}{\partial x} + v_s\frac{\partial S_2}{\partial y}$$

$$w_6 = \frac{1}{\Delta t}S_3 + S_3\frac{\partial u_s}{\partial x} + u_s\frac{\partial S_3}{\partial x} + v_s\frac{\partial S_3}{\partial y}$$

Multiplying w_4 with the residual of Eqn (9.26) we can get the entries of the 4th row of the elemental system. Examples are

$$K_{11}^{2\ e} = \int_{\Omega^e} w_4\left(-S_1\frac{1}{(\rho_s)^2}\frac{\partial p_s}{\partial x}\right)d\Omega$$

$$K_{1,12}^{2\ e} = \int_{\Omega^e} w_4\left(\frac{1}{\rho_s}\frac{\partial S_3}{\partial x}\right)d\Omega$$

Similarly Eqns (9.11c) and (9.11d) can be used to come up with the remaining entries of the elemental system.

9.5 Boundary Conditions

Euler equations contain 1st order derivatives only and no integration by parts is applied. Actually in LSFEM integration by parts is never applied, instead high order derivatives are reduced to lower ones by the definition of new variables. But for Euler equations this is not necessary. Without integration by parts, LSFEM does not have any boundary integral term and therefore it does not support any natural or mixed BCs, but works only with essential BCs. This seems to be a restriction, but actually it is not.

Euler equations model inviscid flows and no slip boundary condition can not be used with them. Instead "no penetration BC" is applied at solid walls, i.e. the velocity vectors should be tangent to solid walls or in other words velocity component perpendicular to a solid wall should be zero. In practice this BC is more difficult to code compared to the no-slip BC. Two commonly used techniques to implement no penetration BC are the penalty method and coordinate rotation method, details of which can be found in reference [1].

Other than the solid walls, typical boundaries used in compressible Euler solutions are inflows and outflows. At inflows all nodal unknowns are specified and in general no BC is specified at outflows.

9.6 Adaptive Mesh Refinement for Shock Capturing

One important difficuty in the solution of compressible flows is the proper capturing of shock waves, which are very thin regions of the flow field with very high gradients. To resolve these discontinuities accurately very small elements are required. Since in general it is not possible to know the exact locations and strengths of the shock waves of a flow field, proper mesh generation is not easy. Solution based Adaptive Mesh Refinement (AMR) becomes a necessity to automatically refine (and coarsen if necessary) the mesh according to the formation of shock waves.

A number of different ideas can be used to determine the elements that needs to be refined. One common approach is to calculate a representative error value for each element as follows

$$err = A^e |\nabla p| \tag{9.27}$$

which is based on the area weighted pressure gradient (change of pressure) over the elements. Error values calculated for each element are normalized using the average value of the errors over all elements and than they are compared against a user specified tolerance value. Elements for which the normalized error value exceeds the limit are refined.

Different strategies can be used to divide the elements that are labeled for refinement. For example triangular elements can be refined by dividing them into two using a line passing through the midpoint of their longest edge and the opposite corner. But as seen in Figure 9.2, such a refinement may result in hanging nodes (red node shown in the figure), which should be avoided by dividing the negihboring elements properly. In mesh coarsening a number of neighboring elements that are not labeled for refinement needs to be joined to form a single element, which is more difficult to implement in a code.

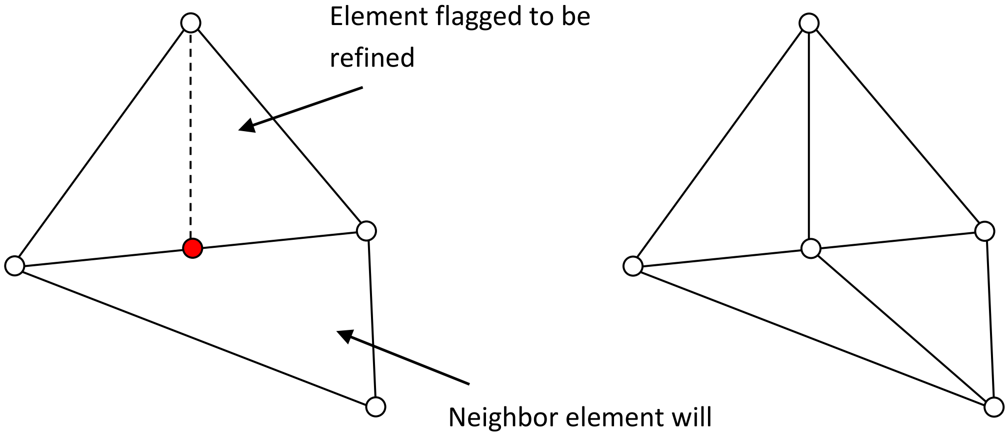


Figure 9.2 Possible element division for triangular elements. Left : Before refienement, Right : After refinement

9.7 Sample Solution – Flow Over a Circular Bump

This is a well known benchmark problem for compresile flow solvers. As seen in Figure 9.3, problem domain consists of a 2D duct with a circular bump at its bottom wall. Bump thichkness is adjusted such that it creates a 4 % blockage. Right boundary is defined as exit and no BC is specified there so that the flow can leave freely. Top and bottom boundaries are specified as solid walls. At the left boundary flow variables are specified as shown, so that inlet Mach number turns out to be 1.65 ($k = 1.4$ is used for air). It is known that this supersonic incoming flow will interact with the bump and an oblique shock wave will emanate from the leading edge of the bump, which will reflect from the upper wall and will further interact with the oblique shock that will originate from the trailing edge of the bump.

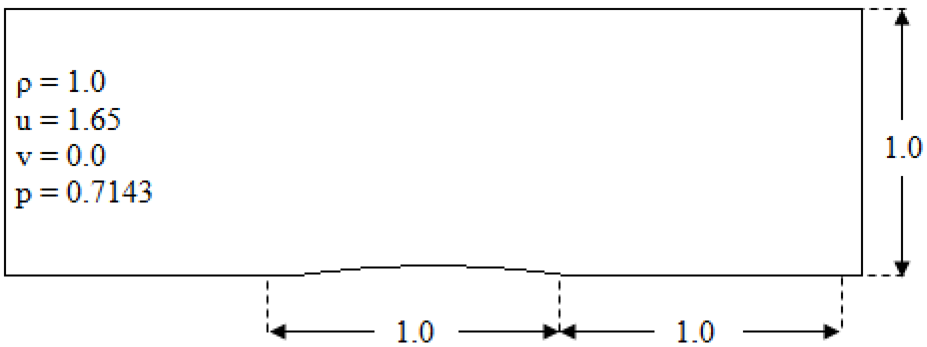


Figure 9.3 Flow over a circular arc bump problem

An adaptive solution is performed starting from the almost uniform mesh shown in Figure 9.4. After a number of adaptive refinement cycles the final mesh generated is also shown in the same figure. It is clearly seen that only the shock locations are refined. Figure 9.5 shows a zoomed view of the elements of the adapted mesh surrounding a shock wave.

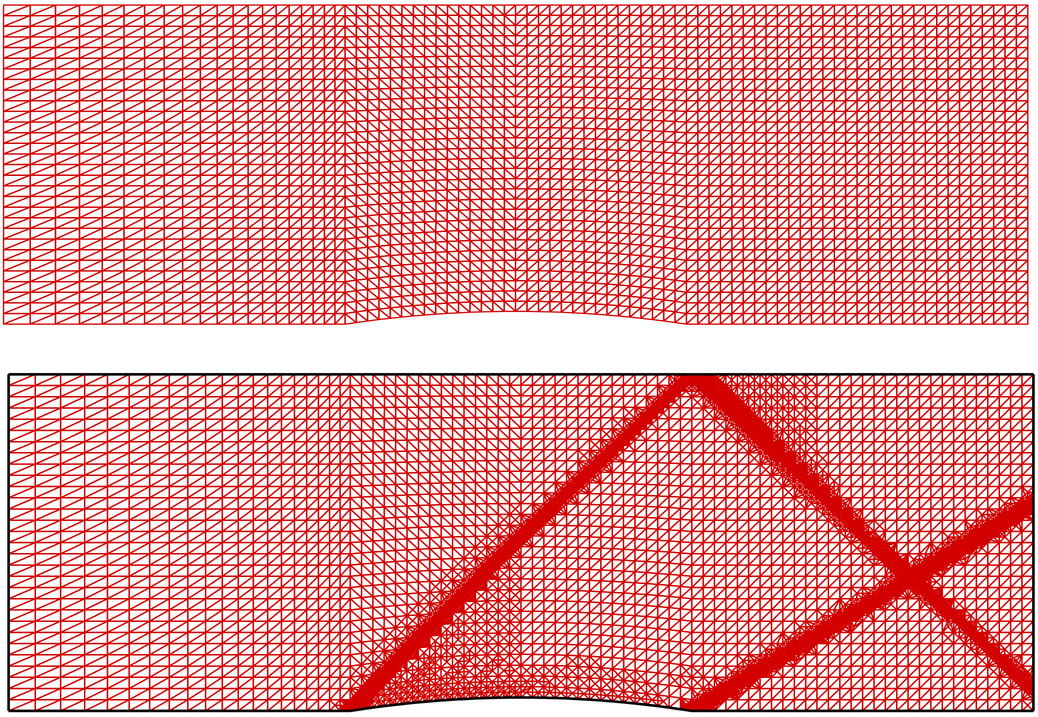


Figure 9.4 Initial and final (adapted) mesh for the flow over a circular bump problem

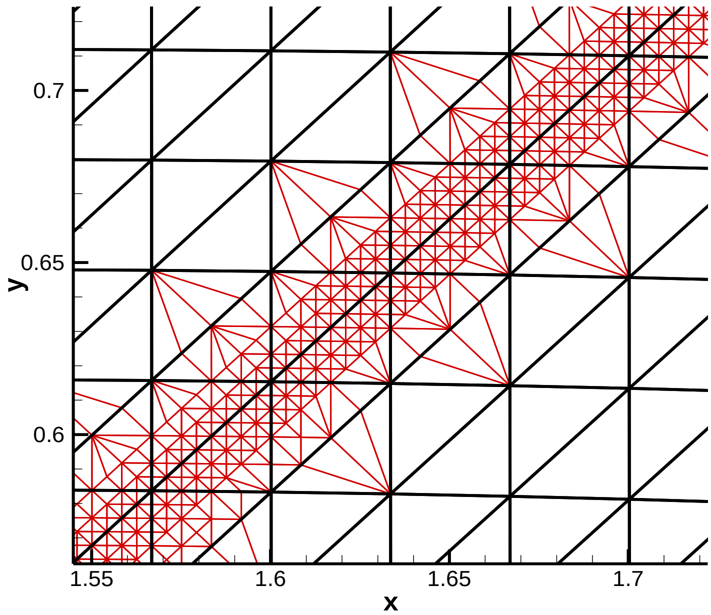


Figure 9.5 Zoomed view of the elements of the adapted mesh surrounding a shock wave. Black thick lines show the initial mesh.

Pressure contours obtained with the final adapted mesh is shown in Figure 9.6. Note that pressure magnitude is nondimensionalized using the density and speed of sound at the inlet.

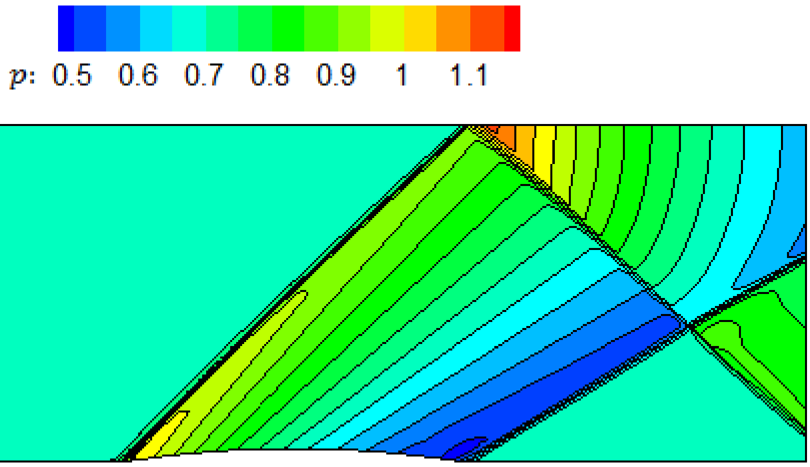


Figure 9.6 Pressure contours for the flow over a circular bump problem obtained using the fian ladapted mesh

9.8 Exercises

E-9.1. In Section 9.4 only a couple entries of $[K^e]$ and $\{F^e\}$ are determined. Calculate the full elemental stiffness matrix and force vector.

E-9.2. Flows over bullets, projectiles, missiles, etc. can effectively be solved as 2D using an axisymmetric formulation. For this, following equation set should be used instead of Eqn (9.3). Determine how the differential operators A_0, A_1 and A_2 will change compared to the planar 2D case. Obtain the elemental stiffness matrix and force vector.

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_z}{\partial z} + v_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial v_r}{\partial r} + v_r \frac{\partial \rho}{\partial r} + \frac{\rho v_r}{r} = 0$$

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} + v_r \frac{\partial v_z}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\frac{\partial v_r}{\partial t} + v_z \frac{\partial v_r}{\partial z} + v_r \frac{\partial v_r}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$\frac{\partial p}{\partial t} + \gamma p \frac{\partial v_z}{\partial z} + v_z \frac{\partial p}{\partial z} + \gamma p \frac{\partial v_r}{\partial r} + v_r \frac{\partial p}{\partial r} + \gamma \frac{p v_r}{r} = 0$$

9.9 References

[1] H. Y. Akargün, Least-Squares Finite Element Solution of Euler Equations with Adaptive Mesh Refinement, Masters Thesis, Dept. of Mechanical Engineering, METU, 2012.